

PHASE TRANSITIONS FOR GEODESIC FLOWS AND THE GEOMETRIC POTENTIAL

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ABSTRACT. In this paper we discuss the phenomenon of phase transitions for the geodesic flow on some geometrically finite negatively curved manifolds. We define a class of potentials *going slowly to zero through the cusps of X* for which, modulo taking coverings, the pressure map $t \mapsto P(tF)$ exhibits a phase transition. By careful choice of the metric at the cusp we can show that the geometric potential (or unstable jacobian) F^{su} belongs to this class of potentials (modulo an additive constant). This results in particular apply for the geodesic flow on a M -puncture sphere for every $M \geq 3$.

1. INTRODUCTION

In this paper we will study the Ergodic theory of geodesic flows on complete negatively curved Riemannian manifolds by means of the *pressure map*. Loosely speaking: if F is a real function on the unit tangent bundle of a manifold X , then its *pressure* is the exponential growth of the number of closed geodesics (counted with ‘multiplicity’). The ‘multiplicity’ associated to each periodic orbit is given by the integral of F with respect to the probability measure that equidistribute the orbit. More precise definitions are given in Section 2. We will consider *potentials* (for historical reasons we call functions on phase space as potentials) with certain regularity, for our purposes Hölder continuity will be enough. We denote by P the map that assign to a Hölder continuous potential F its pressure $P(F)$. We attempt to describe this map. Global information about P is very difficult to obtain, a first approximation to this problem is to restrict P to lines passing through the origin, i.e. restrict our attention to $t \mapsto P(tF)$. This choice of lines is not done by accident, the zero potential has a very particular role in this theory, $P(0)$ is the *topological entropy* of the geodesic flow on X . A sustancial work in this field has been recently done by F. Paulin, M. Pollicot and B. Schapira [PPS]. They studied the pressure of the geodesic flow on negatively curved manifolds and its equilibrium states (without any compactness assumption). They were able to extends a vast number of theorems known for the geodesic flow on compact negatively curved manifolds and its entropy.

Before stating our results we need to introduce some notation. The geodesic flow on X will be denoted by $(g_t)_{t \in \mathbb{R}}$. We denote by $\mathcal{M}(g)$ to the space of probability measures invariants by the geodesic flow and $h_\mu(g)$ denotes the measure theoretic entropy of $\mu \in \mathcal{M}(g)$ under g_1 . It is proven in [PPS] that for a Hölder potential F , its the pressure $P(F)$ satisfies the *variational principle*:

$$P(F) = \sup_{\mu \in \mathcal{M}(g)} \{h_\mu(g) + \int F d\mu : \int F d\mu > -\infty\}.$$

A measure μ is called an *equilibrium measure* of F if it realizes the supremum above, i.e. $P(F) = h_\mu(g) + \int F d\mu$. Geodesic flows on negatively curved manifolds are probably the best understood example of Anosov flows, as such, they inherit all the general features of Anosov flows. In a landmark paper, R. Bowen and D. Ruelle [BR] proved that for Axiom A flows the pressure map $t \mapsto P(tF)$ is real analytic for regular potentials. In fact, every regular potential has an unique equilibrium state. The heart of the argument lies in the use of a symbolic model available for those systems. We will say that F exhibits a *phase transition* if there is a significant change in the dynamics of the geodesic flow with respect to tF , as t varies. A reasonable way to detect such a change is by the existence (or not) of an equilibrium state for the potential tF . In this paper a phase transition of F will be understood as a time t_0 where tF has an equilibrium state for t slightly bigger than t_0 , but not for slightly smaller times (or viceversa). A major problem in this context is to relate changes in the differentiability of the pressure map with changes in the dynamics. For discussions about this problem we refer the reader to [S], [PZ], [IJ],[BI] and references therein. Unfortunately, without a symbolic model to our disposal, strong differentiability properties (e.g. real analyticity) of the pressure are not available. This is the main reason of our definition of a phase transition. As far as the authors know, the modular surface [IJ] and the cases covered in [IRV] are the only examples where phase transitions are known to exist for the geodesic flow. Those phase transitions are achieved by the use of symbolic dynamics, which despite of having some geometric interpretation, have the disadvantage of losing track of important geometric information. In particular, our application to the geometric potential (see Theorem B below) does not seem likely to fit well with symbolic methods.

From now on we will always assume that X is *geometrically finite*. In particular the noncompactness of the nonwandering set of the geodesic flow is controlled by the cusps of X . In this paper we present a geometrical way to create phase transition. We will restrict our attention to potentials which goes to zero through the cusps of T^1X . A fundamental observation is that the existence or not of phase transitions is strongly related with the velocity on which F goes to zero over the ends of the manifold (see Remark 4). It turns out that potentials that do not exhibit phase transitions are easier to construct, we can reduce this computation to terms that only involve the cusps of X (see Remark 5). In this paper we will construct phase transitions on very simple geometrically finite manifolds with exactly one cusp. Its fundamental group is the free product of two groups, this allows us to estimate the critical exponents of the relevant Poincaré series (see Proposition 1). It is an interesting question whether or not phase transitions can occur on any noncompact negatively curved manifold, even when we restrict to the geometrically finite case. For any subgroup G of $Is(\tilde{X})$ one can define its critical exponent δ_G (see Section 2). Using the unique (up to conjugation) maximal parabolic subgroup \mathcal{P} of $\pi_1(X)$, we define $t_F = \sup\{t : P(tF) = \delta_{\mathcal{P}}\}$. We now state one of the main results of this paper.

Theorem A *There exists a geometrically finite negatively curved manifold X with a single cusp such that the following holds. For every nonnegative Hölder continuous potential F going slowly to zero through the cusp of X we have (modulo taking coverings) that*

- (1) $t_F \in [-1, 0)$,
- (2) The potential tF has equilibrium measure for $t > t_F$,
- (3) The potential tF has not equilibrium measure for $t < t_F$.

In other words, the map $t \mapsto P(tF)$ exhibits a phase transition at t_F . Moreover $t \mapsto P(tF_n)$ is differentiable for $t \neq t_{F_n}$.

In Section 3 we define the class of potentials \mathcal{F}_s which ‘go slowly to zero’ through the cusp of X . The function F in Theorem A belongs to the class \mathcal{F}_s . Besides introducing this family of potentials that exhibit phase transitions, we construct a manifold for which the *geometric potential* (or unstable jacobian) exhibits a phase transition. This is done by a careful modification of the metric at the cusp of X . The main difficulty of the construction is that we can not modify the geometric potential without modifying the entire Poincaré series. In Section 4 we prove that the geometric potential of this manifolds belong to \mathcal{F}_s . We remark that it is proved in [PPS] that if the Liouville measure is conservative for the geodesic flow and Ruelle’s inequality holds, then the Liouville measure is proportional to the equilibrium state of the geometric potential. Recently F. Riquelme [R] proved Ruelle’s inequality for the geodesic flow on noncompact negatively curved manifolds. In particular if X has finite volume, then the equilibrium state of the geometric potential is the Liouville measure. We have the following result.

Theorem B *There exists an extended Schottky manifold (X, g) for which the geodesic flow has a measure of maximal entropy and the geometric potential exhibits a phase transition.*

The following remark gives a very concrete family of manifolds for which our results apply (see Corollary 2 and Corollary 3).

Remark 1. *In Theorem A and Theorem B the manifold X can be assumed to be homeomorphic to a M -punctured sphere for any $M \geq 3$. In both cases exactly one puncture is a cusp. In Theorem A we can moreover assume that the metric on X is hyperbolic. In Theorem B the metric is hyperbolic away from a neighborhood of the cusp.*

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2. PRELIMINARIES

2.1. Pressure and entropy for geodesic flows.

In this and following sections (\tilde{X}, g) will always be a simply connected complete negatively curved Riemannian manifold. We assume the pinching condition $-a^2 \leq K_g \leq -b^2$, where $a \geq b > 0$, and K_g is the sectional curvature of the metric g . We will moreover assume that the derivatives of the sectional curvature are uniformly bounded. This assumption is fundamental to use the results in [PPS]. We denote by $\partial_\infty \tilde{X}$ to the visual boundary of \tilde{X} . Let Γ be a discrete, torsion free group of isometries of \tilde{X} and let $X = \tilde{X}/\Gamma$. We denote by π the canonical projection from \tilde{X} to X . The metric g descends to X , for convenience we still denote this metric by

g . The set of vectors with g -norm 1 is denoted by T^1X and $(g_t)_{t \in \mathbb{R}}$ is the geodesic flow on T^1X . The geodesic flow on T^1X is an Anosov flow, i.e. it has strong expanding/contracting properties at the level of tangent spaces. The nonwandering set of the geodesic flow will be denoted by $\Omega \subset T^1X$. The projection of Ω to X is the convex core $C(X)$ of X , this is the smallest convex set in X such that the inclusion $C(X) \hookrightarrow X$ is a homotopy equivalence. By Poincaré recurrence, the support of every (g_t) -invariant probability measures lies in Ω . The action of any isometry on \tilde{X} extends to an homeomorphism of $\partial_\infty \tilde{X}$. The *limit set* of Γ is the smallest closed subset of $\partial_\infty \tilde{X}$ invariant by the action of Γ , this set is denoted by $L(\Gamma)$. In this paper μ will always stand for a (g_t) -invariant probability measure and its measure theoretic entropy is denoted by $h_\mu(g)$. The space of (g_t) -invariant probability measures is denoted by $\mathcal{M}(g)$. We will be mostly interested in noncompact spaces, because of this we need to clarify the notion of convergence of measures that we will be using. We say that a sequence of measures $\{\eta_n\}_{n \in \mathbb{N}}$ *converges vaguely* to a measure η if for every compactly supported continuous function f we have

$$\lim_{n \rightarrow \infty} \int f d\eta_n = \int f d\eta.$$

We remark that the limit measure of a convergent sequence of probability measures might not be a probability measure, we could have lost mass.

Definition 1. *We say that X is geometrically finite if an ϵ -neighborhood of $C(X)$ has finite Liouville measure. We say that Γ is geometrically finite if \tilde{X}/Γ is geometrically finite.*

In Section 3 we will need some important consequences of the geometrically finite assumption. We remark that geodesic flows are very special, even in the family of Anosov flows, as the theorems below will show. We start with a concept that was already mentioned in the introduction.

Definition 2. *Let G be a discrete group of isometries of (\tilde{X}, g) . We define the Poincaré series associated to G based at $x \in \tilde{X}$ as*

$$P(s) = \sum_{\gamma \in G} \exp(-sd(\gamma x, x)).$$

The critical exponent of G is defined as

$$\delta_G = \inf\{s \mid P(s) \text{ is finite}\}.$$

We say that Γ is of convergence type if $P(\delta_\Gamma) < \infty$ and of divergence type if the Poincaré series diverges at δ_Γ .

Observe that the critical exponent δ_Γ does not depend on the base point x and that because of the uniform bounds on the sectional curvature it is a finite number. A general procedure due to S. Patterson and D. Sullivan allows us to associate to Γ a family of conformal measures on $\partial_\infty \tilde{X}$ of exponent δ_Γ , the so called *Patterson-Sullivan conformal measures* of Γ . Using this family of conformal measures one can canonically construct a (g_t) -invariant measure, we refer to this measure as the Bowen-Margulis measure. The importance of the Bowen-Margulis measure and of the critical exponent δ_Γ lies in the following theorem.

Theorem 1. [OP] *The topological entropy of the geodesic flow on $X = \tilde{X}/\Gamma$ is equal to δ_Γ . If the Bowen-Margulis measure is finite, then its normalization is the*

unique probability measure of maximal entropy. If the Bowen-Margulis measure is infinite, then there is not measure of maximal entropy. Moreover, the topological entropy is realized by the Riemannian distance on T^1X .

For more information about the Bowen-Margulis measure and its characteristic properties we refer the reader to [OP]. We also remark that, since we are mainly working in the noncompact setting, whenever we say ‘topological entropy’ we actually mean

$$h_{top}(g_1) = \inf\{h_d(g_1) \mid d \text{ a compatible metric on } T^1X\},$$

where $h_d(g_1)$ is Bowen’s definition of topological entropy for noncompact metric spaces [B]. The variational principle holds for this definition of topological entropy if the ambient space is locally compact (see [HK]). In other words we have

$$\delta_\Gamma = \sup_{\mu \in \mathcal{M}(g)} h_\mu(g).$$

Given two points $x, y \in \tilde{X}$, we will denote by $[x, y]$ to the oriented geodesic segment starting at x and ending at y . For a function $G : T^1\tilde{X} \rightarrow \mathbb{R}$, we will constantly use the notation $\int_x^y G$ to represent the integral of G over the tangent vectors to the path $[x, y]$. Given a function $F : T^1X \rightarrow \mathbb{R}$, we denote by $\tilde{F} : T^1\tilde{X} \rightarrow \mathbb{R}$, to its lift to the universal cover of X . In other words $\tilde{F} = F \circ \tilde{\pi}$, where $\tilde{\pi}$ is the canonical projection $\tilde{\pi} : T^1\tilde{X} \rightarrow T^1X$. Recently F. Paulin, M. Pollicott and B. Schapira developed a successful theory about equilibrium states for Hölder potentials. Analogous to the definition of critical exponent of Γ we have one for pairs (Γ, F) .

Definition 3. Let $F : T^1X \rightarrow \mathbb{R}$ be a continuous function and \tilde{F} its lift to $T^1\tilde{X}$. Define the Poincaré series associated to (Γ, F) based at $x \in \tilde{X}$ as

$$P(s, F) = \sum_{\gamma \in \Gamma} \exp\left(\int_x^{\gamma x} (\tilde{F} - s)\right).$$

The critical exponent of (Γ, F) is

$$\delta_\Gamma^F = \inf\{s \mid P(s, F) \text{ is finite}\}.$$

We say that the pair (Γ, F) is of convergence type if $P(\delta_\Gamma^F, F) < \infty$, in other words the Poincaré series converges at its critical exponent. Otherwise we say (Γ, F) is of divergence type.

If F is Hölder continuous, then the critical exponent does not depend on the base point x . Observe that if F is bounded, then δ_Γ^F is finite. This observation is one of the main reasons why in most of the statements of this paper we assume F to be bounded, in general the hypothesis $\delta_\Gamma^F < \infty$ will be enough. As usual, the pressure of a potential F is defined by

$$P(F) = \sup_{\mu \in \mathcal{M}(g)} \{h_\mu(g) + \int F d\mu : \int F d\mu > -\infty\}.$$

We say that a measure $\mu \in \mathcal{M}(g)$ is an *equilibrium state* for F if $P(F) = h_\mu(g) + \int F d\mu$. Similar to the construction of the Patterson-Sullivan conformal measures, we can associate to the pair (Γ, F) a family of conformal measures. We briefly recall this construction. A *Patterson density* of dimension δ for (Γ, F) is a family of finite

Borel measures $(\sigma_x)_{x \in \tilde{X}}$ on $\partial_\infty \tilde{X}$, such that, for every $\gamma \in \Gamma$, for all $x, y \in \tilde{X}$ and for every $\xi \in \partial_\infty \tilde{X}$ we have

$$\gamma_* \sigma_x = \sigma_{\gamma x} \quad \text{and} \quad \frac{d\sigma_x}{d\sigma_y}(\xi) = e^{-C_{F-\delta, \xi}(x, y)},$$

where $C_{F, \xi}(x, y)$ is the *Gibbs cocycle* defined as

$$C_{F, \xi}(x, y) = \lim_{t \rightarrow \infty} \int_y^{\xi_t} \tilde{F} - \int_x^{\xi_t} \tilde{F},$$

for any geodesic ray $t \mapsto \xi_t$ ending at ξ . Note that the limit in the definition of the Gibbs cocycle always exists since the manifold has negative curvature and the potential is Hölder-continuous. If $\delta_\Gamma^F < \infty$, then there exists at least one Patterson density of dimension δ_Γ^F for (Γ, F) , which support lies in the limit set $L(\Gamma)$ of Γ . If (Γ, F) is of divergence type then there is a unique Patterson density of dimension δ_Γ^F . Using this family of Patterson densities one can construct a (g_t) -invariant measure m_F . We call this measure the *Gibbs measure associated to the potential F* and it is denoted by m_F . A fundamental property of m_F is that, whenever finite, it is the unique equilibrium measure for the potential F . The following theorem is a generalization of Theorem 1.

Theorem 2. [PPS] *Let $F : T^1 X \rightarrow \mathbb{R}$ be a bounded Hölder potential. Then*

$$P(F) = \delta_\Gamma^F.$$

Moreover, if there exists a finite Gibbs measure m_F for (Γ, F) , then $m_F / \|m_F\|$ is the unique equilibrium state of F . Otherwise there is not equilibrium measure.

We stated the theorem for the cases that we will need, for a more complete version of this result and a precise formula for m_F see [PPS, Section 6]. If G is a group, we will use the notation G^* for $G \setminus \{id\}$.

Definition 4. *Let F_1 and F_2 discrete, torsion free subgroups of $\text{Iso}(\tilde{X})$. We say that F_1 and F_2 are in Schottky position if there exist disjoint closed subsets U_{F_1} and U_{F_2} of $\partial_\infty \tilde{X}$ such that $F_1^*(\partial_\infty \tilde{X} \setminus U_{F_1}) \subset U_{F_1}$ and $F_2^*(\partial_\infty \tilde{X} \setminus U_{F_2}) \subset U_{F_2}$.*

The following theorem was proven by Coudène in [Cou] for his definition of critical exponent and reproved in [RV] for the definition of critical exponent used in [PPS]. This provides a useful criterion for a critical gap with respect to the parabolic subgroups.

Theorem 3. *Assume (\mathcal{P}, F) is of divergent type for a parabolic subgroup \mathcal{P} . Then*

$$\delta_{\mathcal{P}}^F < \delta_\Gamma^F.$$

We define the *parabolic critical exponent* of Γ as

$$\bar{\delta}_{\mathcal{P}} = \sup_{\mathcal{P}} \delta_{\mathcal{P}},$$

where the supremum runs over the parabolic subgroups of Γ . In the geometrically finite case this supremum is achieved because there are only finitely many conjugacy classes of maximal parabolic subgroups in Γ .

Definition 5. *We say that $F : T^1 X \rightarrow \mathbb{R}$ goes to zero through the cusps of X if for every $\epsilon > 0$, the set $F^{-1}(-\epsilon, \epsilon)$ contains a neighborhood of every cusps in X .*

We now state a result recently obtained in [RV]. This result and its corollary will be particularly useful to describe the pressure map in the proof of Theorem A.

Theorem 4. [RV] *Let (M, g) be a geometrically finite Riemannian manifold with pinched negative sectional curvature. Let (μ_n) be a sequence of ergodic (g_t) -invariant probability measures on T^1M converging to μ in the vague topology. Let F be a continuous potential going to zero through the cusps of T^1M . Then we have*

$$\limsup_{n \rightarrow \infty} (h_{\mu_n}(g) + \int F d\mu_n) \leq \|\mu\| (h_{\mu/|\mu|}(g) + \int F d\mu/|\mu|) + (1 - \|\mu\|) \bar{\delta}_{\mathcal{P}}.$$

In the statement above $|\mu|$ is the total mass of the (g_t) -invariant measure μ . Because we took a sequence of probability measures, any limiting measure μ will have $|\mu| \leq 1$. If μ is the zero measure, then the RHS is understood as $\bar{\delta}_{\mathcal{P}}$.

Corollary 1. *Assume the hypothesis of Theorem 4. Moreover assume the derivatives of the sectional curvature are uniformly bounded. Let F be a nonnegative Hölder continuous potential going to zero through the cusps of T^1M such that $P(F) > \bar{\delta}_{\mathcal{P}}$. Then the potential F has an unique equilibrium measure.*

For Hölder potentials the existence follows from [PPS, Theorem 8.3]. The existence of the equilibrium state follows even if F is only continuous from Theorem 4. For such potentials very little can be said in terms of the theory developed in [PPS]. We finish this subsection with the following definition.

Definition 6. *We say that a potential F exhibits a phase transition at t_0 if there exists $\epsilon > 0$ such that $P(tF)$ has an equilibrium state for $t \in (t_0, t_0 + \epsilon)$, but it hasn't for $t \in (t_0 - \epsilon, t_0)$ (or vice versa). A potential F exhibits a phase transition if it exhibits a phase transition for some $t_0 \in \mathbb{R}$.*

2.2. Extended Schottky groups.

Let N_1, N_2 be two non-negative integers such that $N_1 + N_2 \geq 2$ and $N_2 \geq 1$. Consider N_1 hyperbolic isometries h_1, \dots, h_{N_1} and N_2 parabolic ones p_1, \dots, p_{N_2} satisfying the following conditions:

- (1) For $1 \leq i \leq N_1$ there exists a compact neighbourhood C_{h_i} of the attracting point ξ_{h_i} of h_i and a compact neighbourhood $C_{h_i^{-1}}$ of the repelling point $\xi_{h_i^{-1}}$ of h_i , such that

$$h_i(\partial X \setminus C_{h_i^{-1}}) \subset C_{h_i}.$$

- (2) For $1 \leq i \leq N_2$ there exists a compact neighbourhood C_{p_i} of the unique fixed point ξ_{p_i} of p_i , such that

$$\forall n \in \mathbb{Z}^* \quad p_i^n(\partial X \setminus C_{p_i}) \subset C_{p_i}.$$

- (3) The $2N_1 + N_2$ neighbourhoods introduced in (1) and (2) are pairwise disjoint.
- (4) The elementary parabolic groups $\langle p_i \rangle$, for $1 \leq i \leq N_2$, are of divergence type.

The group $\Gamma = \langle h_1, \dots, h_{N_1}, p_1, \dots, p_{N_2} \rangle$ is a non-elementary free group which acts properly discontinuously and freely on X (see [DP, Corollary II.2]). Such a group Γ is called an *extended Schottky group*. It is proven in [DP] that it is a geometrically finite group. Note that if $N_2 = 0$, that is the group Γ only contains

hyperbolic elements, then Γ is a classical Schottky group and its geometric and dynamical properties are well understood.

2.3. The Geometric potential.

We briefly recall the construction of the geometric potential. For $x, y \in \tilde{X}$ and $\xi \in \partial_\infty \tilde{X}$ we define the *Busemann function* as

$$b_\xi(x, y) = \lim_{t \rightarrow \infty} d(x, \xi_t) - d(y, \xi_t),$$

where $t \mapsto \xi_t$ is any geodesic ray ending at ξ . Pick a reference point $o \in \tilde{X}$. For every $\xi \in \partial_\infty M$ and $s > 0$, denote by $B_\xi(s)$ the *horoball* centered at ξ of height s relative to o , that is

$$B_\xi(s) = \{y \in \tilde{M} : b_\xi(o, y) \geq s\},$$

where $b_\xi(o, \cdot)$ is the Busemann function at ξ relative to o . The family $\{\partial B_\xi(s)\}_{s \in \mathbb{R}}$ foliates \tilde{X} by codimension one hypersurfaces. This family corresponds to the level sets of $b_\xi(o, \cdot)$, each leaf is called a *horosphere* centered at ξ . Suppose we are given a unit vector v , we are going to define the stable and unstable manifolds passing through v . Let $x \in \tilde{X}$ be the base of v and $s_0 \in \mathbb{R}$ such that $x \in \partial B_\xi(s_0)$. The set of vectors $\{\nabla_y b_\xi(o, y)\}_{y \in \partial B_\xi(s_0)}$ points to ξ and are perpendicular to $\partial B_\xi(s_0)$. This defines a Hölder-submanifold of $T^1 \tilde{X}$ passing through v , the so called *strong stable submanifold* at v , this will be denoted by $W^{ss}(v)$. A similar construction defines the *strong unstable submanifold* at v , just do the substitutions $v \mapsto -v$ and $\nabla_y b_\xi(o, y) \mapsto -\nabla_y b_\xi(o, y)$, this will be denoted by $W^{su}(v)$. The strong (un)stable foliation is defined as the union of the strong (un)stable manifolds. One can characterize the points lying in $W^{ss}(v)$ and $W^{su}(v)$ by the following

$$W^{ss}(v) = \{w \in T^1 \tilde{X} : \lim_{t \rightarrow \infty} d(g_t v, g_t w) = 0\}.$$

$$W^{su}(v) = \{w \in T^1 \tilde{X} : \lim_{t \rightarrow -\infty} d(g_t v, g_t w) = 0\}.$$

Observe that for every $\gamma \in \text{Iso}(\tilde{X})$ we have $W^{ss}(\gamma v) = \gamma W^{ss}(v)$, and $W^{su}(\gamma v) = \gamma W^{su}(v)$, it follows from this that both foliations descend to $T^1 X$. It also follows from the definition that $W^{ss}(g_t(v)) = g_t(W^{ss}(v))$ and $W^{su}(g_t(v)) = g_t(W^{su}(v))$, i.e. the dynamics of the geodesic flow preserves the foliations.

Definition 7. We define the geometric potential or unstable jacobian by the formula

$$F^{su}(\xi) = - \left. \frac{d}{dt} \right|_{t=0} \log \det dg_t|_{W^{su}}(\xi),$$

where the determinant of dg_t is computed with respect to orthonormal basis of the unstable subspaces with respect to g .

The study of the geometric potential for (transitive) Anosov diffeomorphisms and flows is a classical subject. In those cases the equilibrium state of this potential corresponds to the SRB measure. We will need the following theorem [PPS, Theorem 7.2].

Theorem 5. Assume \tilde{X} is a simply connected negatively curved manifold with pinching $-a^2 \leq K_g \leq -b^2 < 0$. Moreover assume that the derivatives of the sectional curvature are uniformly bounded. Then the geometric potential F^{su} is Hölder continuous.

It is proven in [K, Theorem 3.9.1] that if (X, g) satisfies the pinching condition $-a^2 \leq K_g \leq -b^2 < 0$, then we have

$$-(N-1)a \leq F^{su}(v) \leq -(N-1)b,$$

where N is the real dimension of X and $v \in T^1X$. In particular if μ is a (g_t) -invariant probability measure, then for $t \geq 0$ we have

$$h_\mu(g) - t(N-1)a \leq h_\mu(g) + t \int F^{su}(v) d\mu(v) \leq h_\mu(g) - t(N-1)b.$$

By taking supremum on the set of invariant probability measures we obtain

$$h_{top}(g) - t(N-1)a \leq P(tF^{su}) \leq h_{top}(g) - t(N-1)b.$$

The following remark inspires the computations done in Section 4.

Remark 2. Let $U := F^{su} + (N-1)$. As mentioned above if

$$-(1 + \frac{1}{N-1}M)^2 \leq K_g \leq -(1 + \frac{1}{N-1}L)^2,$$

on an open set W of X , then

$$-(N-1)(1 + \frac{1}{N-1}M) \leq F^{su}(v) \leq -(N-1)(1 + \frac{1}{N-1}L),$$

for every vectors v with base in W . Equivalently $-M \leq U \leq -L$.

3. PHASE TRANSITIONS

In this section we will prove Theorem A. We start by proving the following modification of [DOP, Theorem C].

Proposition 1. Let \mathcal{P} a parabolic subgroup of $\text{Iso}(\tilde{X})$ and h a hyperbolic isometry such that \mathcal{P} and $H = \langle h \rangle$ are in Schottky position. Denote by Γ_k to the group generated by \mathcal{P} and $\langle h^k \rangle$ and define $X_k = \tilde{X}/\Gamma_k$. Let $F : T^1X \rightarrow \mathbb{R}$ be a bounded Hölder potential for which (\mathcal{P}, F) is of convergence type and $\delta_{\mathcal{P}}^F > 0$. Suppose that $\int_\gamma F = 0$, where γ is the periodic orbit associated to h . Denote by F_n the lift of F to T^1X_n . Then there exists $N_0 > 0$ such that if $n \geq N_0$ we have $\delta_{\mathcal{P}}^F = \delta_{\Gamma_n}^{F_n}$ and (Γ_n, F_n) is of convergence type.

The heart of the proof of [DOP, Theorem C] is the following elementary fact about Hadamard manifolds. Given $D > 0$, there exists $C = C(D) > 0$ such that for every geodesic triangle with vertices x, y, z , and angle at z bigger than D , then we have

$$d(x, y) \geq d(x, z) + d(z, y) - C.$$

We begin with the following analogous inequality when we have potentials.

Lemma 1. [PPS, Lemma 3.4] Assume that F is a Hölder potential with $\|F\|_\infty < M$. Then for every $r > 0$, for all $x, y \in \tilde{X}$ and $\xi \in \mathcal{O}_x B(y, r)$ we have

$$|C_{F, \xi}(x, y) + \int_x^y \tilde{F}| \leq L(r, M),$$

for certain uniform constant $L(r, M)$. Here $\mathcal{O}_x B(y, r) \subset \partial_\infty \tilde{X}$ denote the set of end points of geodesic rays emanating from x that intersects $B(y, r)$.

In the inequality above we are using the Gibbs cocycle defined in Section 2. In the proof of Proposition 1 we will use the fact that $C_{F,\xi}(x, y)$ is an additive cocycle, more precisely that $C_{F,\xi}(x, y) + C_{F,\xi}(y, z) = C_{F,\xi}(x, z)$. We now start the proof of Proposition 1.

Proof. Since H and \mathcal{P} are in Schottky position we can find $U_{\mathcal{P}}, U_H \subset X \cup \partial X$ so that

- (1) $\mathcal{P}^*(\partial X \setminus U_{\mathcal{P}}) \subset U_{\mathcal{P}}$.
- (2) $H^*(\partial X \setminus U_H) \subset U_H$.
- (3) $U_H \cap U_{\mathcal{P}} = \emptyset$.

Fix $x \in X$ over the axis of h so that $x \notin U_H \cup U_{\mathcal{P}}$. As a consequence of the Ping Pong Lemma we have that Γ is isomorphic to the free product $H * \mathcal{P}$. By the comments above Lemma 1 we know that there exists a positive constant C such that for every $y \in U_H$ and $z \in U_{\mathcal{P}}$ we have

$$d(y, z) \geq d(x, y) + d(x, z) - C.$$

Applying inequality (3) and the inclusion properties described above we obtain

$$d(x, p_1 h^{kn_1} \dots p_j h^{kn_j} x) \geq \sum_i d(x, p_i x) + \sum_i d(x, h^{kn_i} x) - 2jC,$$

where $n_i \in \mathbb{Z}^*$, $k \neq 0$ and $p_i \in \mathcal{P}^*$. Let M be a bound for F , i.e. $\|F\|_{\infty} < M$. By picking ξ outside $U_{\mathcal{P}} \cup U_H$ and r big enough we can apply Lemma 1. Then

$$\begin{aligned} \int_x^{h^n x} \tilde{F} + \int_x^{p x} \tilde{F} &= \int_{p x}^{p h^n x} \tilde{F} + \int_x^{p x} \tilde{F} \geq -2L(r, M) - C_{F,\xi}(x, p h^n x) \\ &\geq -3L(r, M) + \int_x^{p h^n x} \tilde{F}. \end{aligned}$$

This immediately generalize to

$$\int_x^{p_1 h^{kn_1} \dots p_j h^{kn_j} x} \tilde{F} \leq (2j+1)L(r, M) + \sum_i \int_x^{h^{kn_i} x} \tilde{F} + \sum_i \int_x^{p_i x} \tilde{F}.$$

Let $l := d(x, h x)$. By the choice of x we have $d(x, h^N x) = |N|l$, and since $\int_x^{h x} \tilde{F} = 0$, we also have $\int_x^{h^N x} \tilde{F} = 0$. Finally

$$\sum_{n \in \mathbb{Z}^*} \exp\left(\int_x^{h^{kn} x} (\tilde{F} - s)\right) = \sum_{n \in \mathbb{Z}^*} \exp(-s|n|lk) = 2 \frac{\exp(-slk)}{1 - \exp(-slk)}.$$

For simplicity we will bound the expression

$$\tilde{P}(s) = \sum_{j \geq 1} \sum_{p_i \in \mathcal{P}^*, m_i \in \mathbb{Z}^*} \exp\left(\int_x^{p_1 h^{kn_1} \dots p_j h^{kn_j} x} (\tilde{F} - s)\right).$$

A bound for $P(s)$ follows identically, but here we have more symmetry. Using the inequalities above we obtain

$$(1) \quad \tilde{P}(s) \leq \sum_{j \geq 1} \left(e^{2(C+L(r,M)+1)} \sum_{n \in \mathbb{Z}^*} \exp\left(\int_x^{h^{kn} x} (\tilde{F} - s)\right) \sum_{p \in \mathcal{P}^*} \exp\left(\int_x^{p x} (\tilde{F} - s)\right) \right)^j.$$

By taking k big enough, we can make the right hand side of (1) to be convergent at $s = \delta_{\mathcal{P}}^F$. In particular $\delta_{\mathcal{P}}^F \geq \delta_{\Gamma_k}^{F_k}$, which immediately implies $\delta_{\Gamma_k}^{F_k} = \delta_{\mathcal{P}}^F$. The convergence of (Γ_k, F_k) follows from the same inequality. \square

Remark 3. Let $\{h_1, \dots, h_l\}$ be a collection of hyperbolic isometries and denote by H_i to the group generated by h_i . Suppose that the subgroups $\{H_1, \dots, H_l, \mathcal{P}\}$ are pairwise in Schottky position (just as in the definition of extended Schottky but allowing \mathcal{P} to have bigger rank). Moreover assume that F vanishes over the closed geodesics associated to the h_i 's. Define Γ_n to be the group generated by \mathcal{P} and the elements $\{h_1^n, \dots, h_l^n\}$. The proof of Proposition 1 can be easily modified to conclude that for big enough k we have

$$\delta_{\mathcal{P}}^F = \delta_{\Gamma_k}^{F_k},$$

and that (Γ_k, F_k) is of convergence type. For simplicity in we will mostly state our results the case treated in Proposition 1, but everything works identically under the hypothesis of this remark.

In what follows we will use the standard model of the ends of geometrically finite manifolds on which the nonwandering set Ω penetrates. A fundamental result of B. Bowditch [Bo] says that the ends of geometrically finite manifolds on which Ω penetrates are standard cusps, i.e. given $o \in \tilde{X}$ and big enough s , a neighborhood of the end is isometric to $B_\xi(s)/\mathcal{P}$, where \mathcal{P} is a maximal parabolic subgroup and ξ its fixed point. Moreover, X has finitely many cusps, or equivalently, finitely many maximal parabolic subgroups (up to conjugation), and $\pi\Omega \cap (B_\xi(s) \setminus B_\xi(s+R))$ is relatively compact in X for every $R > 0$.

Lemma 2. Suppose $X = \tilde{X}/\Gamma$ is geometrically finite. If $F : T^1X \rightarrow \mathbb{R}$ be a Hölder potential going to zero through the cusps. Then for every maximal parabolic subgroup \mathcal{P} of Γ we have $\delta_{\mathcal{P}}^F = \delta_{\mathcal{P}}$.

Proof. Pick a maximal parabolic subgroup \mathcal{P} and consider a point $x \in T^1X$ so that $\pi(x)$ belong to the region where the cusp is standard, i.e. where it looks like $B_\xi(s_0)/\mathcal{P}$. We will moreover assume that x belong to the region of the cusp where $|F| < \epsilon$. Using the convexity of the horoballs we get that the Poincaré series of \mathcal{P} based at x can be bounded below by $\sum_{p \in \mathcal{P}} \exp(-(s+\epsilon)d(x, px))$ and above by $\sum_{p \in \mathcal{P}} \exp(-(s-\epsilon)d(x, px))$. This easily implies that $|\delta_{\mathcal{P}}^F - \delta_{\mathcal{P}}| < 2\epsilon$. Since ϵ was arbitrary we conclude the lemma. \square

Our next proposition is very important in this paper, it provides a family of potentials for which we have phase transitions. In Section 4 we will also use ideas from the proof to modify the metric at the cusp. Recall that $(\mathcal{P}, -F)$ is of convergence type if the sum

$$\sum_{p \in \mathcal{P}} \exp\left(-\int_x^{px} \tilde{F} - \delta_{\mathcal{P}}^{-F} d(x, px)\right),$$

is finite.

Proposition 2. Let Γ be a geometrically finite group of isometries. There exists a bounded Hölder potential F_0 on $T^1X = T^1\tilde{X}/\Gamma$ satisfying the following conditions:

- (1) F_0 is positive near the cusps,
- (2) F_0 goes to zero through the cusps of X ,

- (3) $(\mathcal{P}, -F_0)$ is of convergence type for every maximal parabolic subgroup \mathcal{P} of Γ .

Proof. We will define F in a neighborhood of each cusp, then we extend F to the rest of the manifold in any bounded Hölder continuous way. We pick a maximal parabolic subgroup \mathcal{P} of Γ that fixes $\xi \in \partial_\infty \tilde{X}$. There exists a neighborhood \mathcal{U} of the cusp associated to \mathcal{P} which is isometric to $B_\xi(s_0)/\mathcal{P}$, for big enough s_0 . Pick a reference vector x such that $\pi(x) \in \partial B_\xi(s_0)/\mathcal{P}$ and which points to the interior of the cusp. For $y \in \mathcal{U}$, define $d(y) = s$ if $\pi(y) \in \partial B_\xi(s + s_0)/\mathcal{P}$. We say that the geodesic segment $\gamma : [a, b] \rightarrow X$ in \mathcal{U} has *height* H if

$$\max_{t \in [a, b]} d(\gamma(t)) = H.$$

For $l < L$, define

$$S(l, L) := \{p \in \mathcal{P} : l < d(x, px) \leq L\}.$$

By the definition of critical exponent, for every $\epsilon > 0$, there exists a natural number $C(\epsilon)$ so that

$$\sum_{p \in S(C(\epsilon), \infty)} \exp(-(\delta_{\mathcal{P}} + \epsilon)d(x, px)) < \epsilon^2.$$

We define a sequence $\{A_n\}_{n \in \mathbb{N}}$ inductively as follows. Let $A_1 = C(1)$ and $A_{n+1} = \max(A_n + 1, C(1/n))$. By construction the sequence of natural numbers $\{A_n\}_{n \in \mathbb{N}}$ is strictly increasing and satisfies

$$\sum_{p \in S(A_n, \infty)} \exp(-(\delta_{\mathcal{P}} + 1/n)d(x, px)) < \frac{1}{n^2}.$$

We define H_n to be maximum height of the geodesic segments $[x, px]$ for $p \in S(A_n, A_{n+1})$. With the set of heights H_n we construct a sequence $\{B_n\}_{n \geq 1}$ by declaring $B_1 = H_1$, and inductively define $B_{n+1} = \max(B_n + 1, H_{n+1})$. Define a function f on $\mathcal{V} = d^{-1}([B_1, \infty))$ by the following expression

$$f(x) = \begin{cases} -d(x) + 1/n - B_n & \text{if } x \in d^{-1}([B_n, B_n + 1/n - 1/(n+1)]) \\ 1/(n+1) & \text{if } x \in d^{-1}([B_n + 1/n - 1/(n+1), B_{n+1}]). \end{cases}$$

Let $F : T^1\mathcal{V} \rightarrow \mathbb{R}$ be the composition of the projection from $T^1\mathcal{V}$ to \mathcal{V} and f . We do the same construction for every cusp in X . Using this functions at the cusps and any bounded Hölder continuous extension to the rest of the manifold, we obtain a bounded Hölder continuous function $F_0 : T^1X \rightarrow \mathbb{R}$. It follows from the construction that F_0 goes to zero through the cusps. We then apply Lemma 2 to get that $\delta_{\mathcal{P}}^{-F_0} = \delta_{\mathcal{P}}$. It only remains to check that $(\mathcal{P}, -F_0)$ is of convergence type for every maximal parabolic subgroup of Γ . Observe that if $p \in S(A_n, A_{n+1})$, then $[x, px]$ has height at most H_n , in particular at most B_n . This implies that if $p \in S(A_n, A_{n+1})$, then the value of F_0 along the segment $[x, px]$ is at least $1/n$. In other words for $p \in S(A_n, A_{n+1})$ we have

$$\int_x^{px} F_0 \geq d(x, px)/n.$$

Finally

$$\sum_{p \in S(A_1, \infty)} \exp(-\int_x^{px} F_0 - \delta_{\mathcal{P}} d(x, px)) = \sum_{n=1}^{\infty} \sum_{p \in S(A_n, A_{n+1})} \exp(-\int_x^{px} F_0 - \delta_{\mathcal{P}} d(x, px))$$

$$\begin{aligned}
 &\leq \sum_{n=1}^{\infty} \sum_{p \in S(A_n, A_{n+1})} \exp\left(-\frac{1}{n}d(x, px) - \delta_{\mathcal{P}}d(x, px)\right) \\
 &\leq \sum_{n=1}^{\infty} \sum_{p \in S(A_n, \infty)} \exp\left(-\left(\frac{1}{n} + \delta_{\mathcal{P}}\right)d(x, px)\right) \\
 &\leq \sum_{n=1}^{\infty} \frac{1}{n^2},
 \end{aligned}$$

which is certainly finite. □

Remark 4. Since the numbers $\{B_n\}$ can be arbitrarily far apart, we interpret the decay of F through the cusp associated to \mathcal{P} as ‘very slow’.

Remark 5. In [RV, Example p. 20] it is constructed a potential which does not exhibit a phase transition. This can be done in any geometrically finite manifold with divergence type maximal parabolic subgroups. The relevant manifolds in this paper satisfy this condition, in particular we know the existence of potentials with no phase transitions.

Definition 8. A potential $G : T^1X \rightarrow \mathbb{R}$ belongs to the family \mathcal{F}_s if the following conditions are satisfied.

- (1) G is a bounded Hölder continuous potential,
- (2) G is positive in a neighborhood of the cusps of X and it goes to zero through the cusps of X ,
- (3) $(\mathcal{P}, -G)$ is of convergence type for every maximal parabolic subgroup \mathcal{P} of Γ .

The elements in \mathcal{F}_s are called potentials going slowly to zero through the cusps of X . The class of nonnegative potentials in \mathcal{F}_s is denoted by \mathcal{F}_s^+ .

The family \mathcal{F}_s is not empty because of Lemma 2. We remark that the family \mathcal{F}_s is very big. Let $F \in \mathcal{F}_s$, if G is a potential going to zero through the cusps of X satisfying $G \geq F$ in a neighborhood of the cusps, then $G \in \mathcal{F}_s$. We now proceed to prove Theorem A. The statement of the following result looks a bit more complicated than Theorem A, but has the advantage of being more precise. As in the introduction, for a potential F we define

$$t_F = \sup\{t : \delta_{\Gamma}^{tF} = \delta_{\mathcal{P}}\}.$$

Theorem 6. Let \mathcal{P} be a divergence type parabolic subgroup of $\text{Iso}(\tilde{X})$ and h a hyperbolic isometry such that $\langle h \rangle$ and \mathcal{P} are in Schottky position. Define Γ_n as the group generated by \mathcal{P} and $\langle h^n \rangle$. Let $X_k = \tilde{X}/\Gamma_k$ and $F : T^1X_1 \rightarrow \mathbb{R}$ a potential in the class \mathcal{F}_s^+ . Assume that $\int_{\gamma} F = 0$, where γ is the periodic orbit associated to h . Let F_n be the lift of F to T^1X_n and N_0 as in Proposition 1. Then for $n \geq N_0$ we have:

- (1) $t_{F_n} \in [-1, 0)$.
- (2) The potential tF_n has equilibrium measure for $t > t_{F_n}$.
- (3) The potential tF_n has not equilibrium measure for $t < t_{F_n}$.

In conclusion the map $t \mapsto P(tF_n)$ exhibit a phase transition at t_{F_n} . Moreover $t \mapsto P(tF_n)$ is differentiable for $t \neq t_{F_n}$.

Proof. It follows easily from the definition of the pressure that $t \mapsto P(tF_n)$ is convex. From now on we will assume that $n > N_0$. Since tF_n goes to zero through the cusps of T^1X_n , we obtain using Lemma 2 that $\delta_{\mathcal{P}} = \delta_{\mathcal{P}}^{tF_n}$. Since F_n is nonnegative we have that $t \mapsto \delta_{\Gamma}^{tF_n}$ is monotone nondecreasing and $\delta_{\mathcal{P}} = \delta_{\mathcal{P}}^{tF_n} \leq \delta_{\Gamma}^{tF_n}$. We conclude that for $t > t_{F_n}$ we have $P(tF_n) > \delta_{\mathcal{P}}$ and $P(tF_n) = \delta_{\mathcal{P}}$ for $t \leq t_{F_n}$. It follows from Corollary 1 that for $t > t_{F_n}$ there exists a unique equilibrium state for tF_n . The proof of the differentiability of the pressure map on (t_F, ∞) follows by standard arguments (see Proposition 3 below). Observe that if (\mathcal{P}, tF_n) is of divergence type, then by Theorem 3 we have $\delta_{\mathcal{P}} = \delta_{\mathcal{P}}^{tF_n} < \delta_{\Gamma}^{tF_n}$, in particular $t > t_{F_n}$. By assumption \mathcal{P} is of divergence type, then Theorem 3 gives us $\delta_{\mathcal{P}} < \delta_{\Gamma} = P(0)$. In particular $t_F < 0$. By Proposition 1 we have that $(\Gamma, -F_n)$ is of convergent type and $P(-F_n) = \delta_{\mathcal{P}}$. In particular by Hopf-Tsuji-Sullivan-Roblin theorem there is not equilibrium state for the potential $-F_n$ and therefore $t_{F_n} \geq -1$. We will prove by contradiction that for $t < t_{F_n}$ there is not equilibrium state. Suppose that μ_t is the equilibrium state of tF_n . If $\int F_n d\mu_t$ is a positive number, then

$$P(tF_n) = h_{\mu_t}(g) + t \int F_n d\mu_t < h_{\mu_t}(g) + (t + \epsilon) \int F_n d\mu_t \leq P((t + \epsilon)F_n).$$

This is a contradiction if $t < t_{F_n}$, in that case for small ϵ both sides are equal to $\delta_{\mathcal{P}}$. We conclude that $\int F_n d\mu_t = 0$ and therefore $h_{\mu_t}(g) = \delta_{\mathcal{P}}$. This implies that μ_t is an equilibrium state for sF_n for all $s < t_{F_n}$. This contradicts that $-F_n$ has not equilibrium state. \square

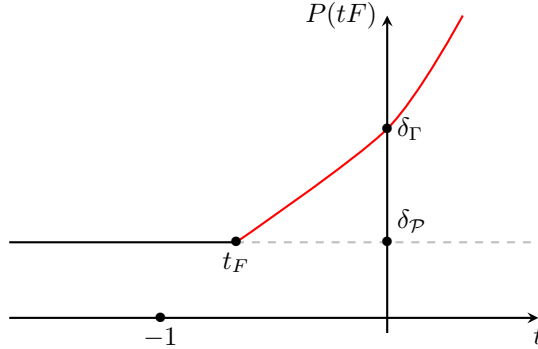


Figure 1: Phase transition for $F \in \mathcal{F}_s^+$

We will briefly discuss what happen if we only assume $F \in \mathcal{F}_s$, i.e. we allow F to take negative values. For this we will need the following fact (see [RV, Proposition 5.8]). For convenience we will assume X has only one cusp.

Proposition 3. *Let F be a continuous potential going to zero through the cusp of X . Assume $P(tF) > \delta_{\mathcal{P}}$ for every $t \in (a, b)$. Then the pressure map is differentiable on (a, b) and for $s \in (a, b)$ we have the formula*

$$\frac{d}{dt} \Big|_{t=s} P(sF) = \int F d\mu_s,$$

where μ_s is the equilibrium state of sF .

Suppose that for every (g_t) -invariant probability measure we have $\int F_n d\mu \geq 0$. Then Proposition 3 and the convexity of the pressure map implies that $t \mapsto P(tF_n)$ has the same description as a potential in \mathcal{F}_s^+ . If there exists a (g_t) -invariant probability measure μ such that $\int F_n d\mu < 0$, then the set

$$J = \{t \in \mathbb{R} : P(tF_n) = \delta_{\mathcal{P}}\},$$

is a compact interval. If $t \in \mathbb{R} \setminus J$, then tF_n has an equilibrium state, and if $t \in \text{int}(J)$ then tF_n has not equilibrium state. Examples like this are easy to construct. For instance suppose that F is negative in the complement of a neighborhood of the cusp and zero on γ . By taking a closed geodesic that wrap around the lift of γ to X_n a sufficiently large number of times (the geodesic represented by ph^{nk} for big k works) we get an invariant measure with negative integral against F_n . Observe that if $F \in \mathcal{F}_s$ and $t > 1$, then $tF \in \mathcal{F}_s$. In particular, given $M > 1$, there exists $N_1 = N_1(M, F)$ such that the following holds. For every $n \geq N_1$, the pairs $(\Gamma_n, -MF_n)$ and $(\Gamma_n, -F_n)$ are of convergence type and have critical exponent $\delta_{\mathcal{P}}$. In particular J contains the interval $(-M, -1)$. An identical modification allows us to construct F such that J contains any bounded open interval in $(-\infty, 0)$. In particular J can be made arbitrarily large.

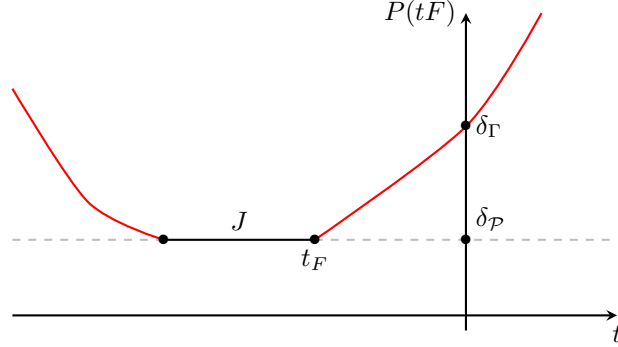


Figure 2: Phase transition for $F \in \mathcal{F}_s$

In light of this discussion we have the following definition.

Definition 9. A potential F going to zero through the cusps of X exhibits a phase transition of type A if the graph of $t \mapsto P(tF)$ looks like Figure 1. A potential F exhibits a phase transition of type B if the graph of $t \mapsto P(tF)$ looks like Figure 2.

The last possible description for the pressure map $t \mapsto P(tF)$ is the reflection of a type A phase transition under the y -axis. Since $F \mapsto -F$ has the same effect on the pressure map, we are basically representing all types of phase transitions for potentials going to zero through the cusps.

Remark 6. The manifolds $\{X_n\}_{n \geq 1}$ constructed in Theorem 6 are all diffeomorphic to X_1 . It is well known that every parabolic subgroup of $\text{Iso}(\mathbb{H}^N)$ is of divergence type. It follows easily that one can verify the hypothesis of Theorem 6 if (\tilde{X}, g) is isometric to \mathbb{H}^N . A concrete situation on which Theorem 6 applies is given in Corollary 2.

Corollary 2. Let X be a thrice-punctured sphere. We can endow X with a complete hyperbolic metric g having exactly one cusp such that it is possible to construct potentials $F : T^1X \rightarrow \mathbb{R}$ exhibiting phase transitions as described in Theorem 6 or

in the paragraph above. By Remark 3 the same holds if X is a M -punctured sphere and $M \geq 3$.

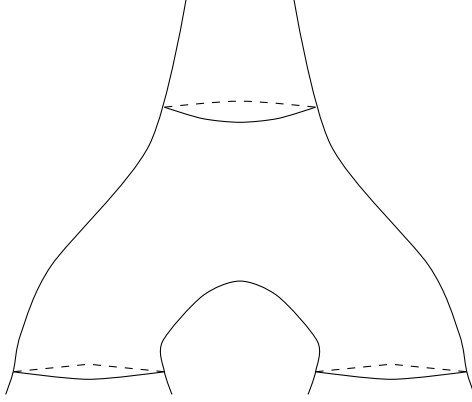


Figure 3: Thrice-puncture sphere with one cusp

4. MODIFICATION AT THE CUSP AND PHASE TRANSITIONS

In this section we will modify the metric at the cusp of a hyperbolic manifold in such a way that the geometric potential does not have an equilibrium state. Since F^{su} is not a potential that goes to zero through the cusps of the manifold, to apply the previous techniques we need to consider a normalization. From now on we will assume that our manifold has real dimension N .

Definition 10. We define the normalized unstable jacobian as the function

$$U = F^{su} + (N - 1).$$

Our goal is to prove that under certain conditions U exhibits a phase transition, just as the potentials in Theorem 6. The construction starts with hyperbolic space \mathbb{H}^N . It is convenient to think in the half space model, i.e. the space $\mathbb{R}^{N-1} \times \mathbb{R}^+$ with coordinates $(x_1, \dots, x_{N-1}, x_0)$ and metric

$$ds^2 = \frac{1}{x_0^2} (dx_1^2 + \dots + dx_{N-1}^2 + dx_0^2).$$

To simplify notation we denote $(x_1, \dots, x_{N-1}, x_0) = (\mathbf{x}, x_0)$. In this model we have a preferent point at infinity, we denote this by $\xi_\infty \in \partial_\infty \tilde{X}$. We will modify the hyperbolic metric in a neighborhood of ξ_∞ . It will be convenient to consider the diffeomorphism $\mathbb{H}^N \rightarrow \mathbb{R}^N$ taking (\mathbf{x}, x_0) to $(\mathbf{x}, \log(x_0))$. In this model the hyperbolic metric takes the form $g = e^{-2t} d\mathbf{x}^2 + dt^2$. For a positive function $T : \mathbb{R} \rightarrow \mathbb{R}$ we define the Riemannian metric

$$g_T = T(t)^2 d\mathbf{x}^2 + dt^2.$$

The sectional curvature of g_T has value $-(T''(t)/T(t))$ for the planes generated by $\langle \partial/\partial x_i, \partial/\partial x_j \rangle$ and value $-(T''(t)/T(t))^2$ for those generated by $\langle \partial/\partial x_i, \partial/\partial t \rangle$. Define

$$K(t) = -(T''(t)/T(t)).$$

Bounds on $K(t)$ clearly imply bounds on the curvature of g_T . The lines $t \mapsto (\mathbf{x}, t)$ are still geodesics and any isometry of \mathbb{R}^{N-1} acts isometrically on (\mathbb{H}^N, g_T) , where the action is given by $A.(\mathbf{x}, t) = (A(\mathbf{x}), t)$. Observe that translations act transitively

in $H_t = \{(\mathbf{x}, t) : \mathbf{x} \in \mathbb{R}^{N-1}\}$. This two basic observations and the definition of the Busemann function are enough to conclude that H_t are the horospheres associated to ξ_∞ . For a function constant on horospheres will use the notation $F(t) = F(\mathbf{x}, t)$. In this section the *height of a segment* will be the maximum value of the t coordinate over the segment. Suppose we have $u : (0, \infty) \rightarrow \mathbb{R}$ a surjective, strictly increasing function. We can define $T = T(u)$ by the equation

$$T(u(t)) = 1/t.$$

In this context we will use $o = (\mathbf{0}, u(1))$ as reference point. Let p be a translation in \mathbb{R}^N such that $d(o, po) = 1$. It is proven in [DOP, Section 3] that there exists a uniform constant C such that

$$|d_T(o, p^n o) - 2u(|n|)| \leq C.$$

It will be important for us the fact that C only depends on the pinching of the sectional curvature of (\mathbb{H}^N, g_T) . Using the symmetry of our metric we can also say that

$$|t_n - u(|n|)| \leq D,$$

where t_n is the maximum height of the geodesic segment $[o, p^n o]$. As before, D only depends on the pinching of the metric. From now on D and C are the constants associated to a metric with pinching $-(1/3)^2 \geq K_g \geq -2^2$. Observe that

$$\begin{aligned} K(u(t)) &= -\frac{2tu'(t) + t^2u''(t)}{(u't)^3}, \\ &= -\frac{1 + 2t\varphi'(t) + t^2\varphi''(t)}{(1 + t\varphi'(t))^3}, \\ &= -\frac{g_1(t) + tg_1'(t)}{g_1(t)^3}, \\ &= -(g_2(t) - \frac{t}{2}g_2'(t)), \end{aligned}$$

where we have made the substitutions

$$u(t) = \log(t) + \varphi(t); g_1(t) = 1 + t\varphi'(t) \text{ and } g_2(t) = 1/g_1(t)^2.$$

4.1. Construction of a special metric at the cusp.

We will start by constructing a function g_2 satisfying several properties. As above, this will give us a function u and therefore a metric on \mathbb{R}^N .

Let $a_{n+1} := (1 + \frac{1}{(N-1)n})^2$, $b_n := 1 - (2a_n - 1)^{-\frac{1}{2}}$ and $c_n := (a_{n-1} - a_n)$. For $n \geq 2$ we choose $k(n) \in \mathbb{N}$ such that

$$\exp\left(\left(\frac{1}{n} + \frac{1}{2}\right)D\right) \sum_{|k| \geq k(n)} \exp\left(-\left(\frac{1}{2} + \frac{1}{n}\right)2\log(|k|)\right) \leq \frac{1}{n^2}.$$

Without loss of generality we can assume that $\{k(n)\}_{n \in \mathbb{N}}$ is a strictly increasing sequence. We will define a sequence $\{p_n\}$ so that the conditions below are satisfied. We do this by induction, i.e. the choice of p_1, \dots, p_n will determine p_{n+1} . For $n \geq 2$ define $\Delta_n = p_n - p_{n-1}$.

- (1) The sequence $\left(\frac{c_n}{\Delta_n}\right)$ is a strictly decreasing.
- (2) $b_n \log(p_{n+1}) \geq -b_n \log(p_n) + \sum_{i=2}^{n-1} b_i \log(p_{i+1}/p_i)$, for $n \geq 3$.
- (3) $(1 - 2b_n) \log(p_{n+1}) \geq \log(k(n+1))$, whenever $1 - 2b_n > 0$.

$$(4) \lim_{n \rightarrow 0} \frac{p_n c_n}{\Delta_n} = 0.$$

For $n \geq 2$ define the line connecting the points (p_n, a_n) and (p_{n+1}, a_{n+1}) as J_n . We could have assumed that p_2 is big enough compared to p_1 so that $J_2(0) \leq 2$, we will do assume that. Define the intervals $I_n = [p_{n-1}, p_n]$. We will construct a \mathcal{C}^∞ function $g_2 : \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfying the following properties

- (1) For $t \in I_n$ and $n \geq 3$ we have $2a_{n-1} - 1 \geq g_2(t) \geq a_n$.
- (2) $g_2''(t) \geq 0 \geq g_2'(t)$ for $t \geq 2$.
- (3) $2 \geq g_2(t) - \frac{t}{2}g_2'(t) \geq 1/3$ for all $t \in \mathbb{R}^+$.
- (4) For $t \in I_n$ and $n \geq 3$ we have $g_2(t) - \frac{t}{2}g_2'(t) \geq a_n$.
- (5) $g_2(t) = 1$ for $t < 1$.

Observe that $(g_2 - (t/2)g_2')' = g_2'/2 - (t/2)g_2''$, so the condition $g_2'' \geq 0 \geq g_2'$ implies that $g_2(t) - (t/2)g_2'(t)$ is non-increasing. If $0 \geq g_2'$, then $g_2(t) - \frac{1}{2}tg_2'(t) \geq g_2(t)$, so if $g_2 \geq a_n$ the same hold for $g_2(t) - \frac{1}{2}tg_2'(t)$. We now explain how to construct g_2 . First define $J_0 : \mathbb{R}^+ \rightarrow \mathbb{R}$ as $J_0(t) := \sup_{n \geq 2} J_n(t)$. Now define

$$J(t) := \begin{cases} 1 & \text{if } t \in (0, 1] \\ \min\{J_0(t), t\} & \text{if } t \geq 1 \end{cases}$$

The function J is not smooth, we can smooth J in a neighborhood of the nodes to obtain a smooth function J^* as close to J (in the \mathcal{C}^∞ topology) as needed. When $t \geq 2$ we can assume that $J^* \geq J$ and that J^* is still convex decreasing. The fact that J^* can be taken convex on that region comes from requirement (1) in the definition of $\{p_n\}$. The choice of $2a_{n-1} - 1$ as the upper bound is just to get room for this perturbation (notice $2a_{n-1} - 1 > a_{n-1}$). We finally set $g_2 = J^*$. Define a function φ (up to additive constant) by the equation

$$\varphi'(t) = (g_2^{-1/2} - 1)/t.$$

First observe that $g_2 \geq 1$ implies that $\varphi' \leq 0$. For $n \geq 3$ and $t \in I_n$ we have $2a_{n-1} - 1 \geq g_2(t)$, then $\varphi'(t) \geq -b_{n-1}/t$. Since $\{b_n\}$ is a positive decreasing sequence we actually have that $\varphi'(t) \geq -b_{n-1}/t$, for every $t \geq p_{n-1}$. Then for $n \geq 3$ and $t \geq p_{n+1}$ we get

$$\begin{aligned} \varphi(t) - \varphi(p_2) &= \sum_{i=2}^{n-1} \int_{p_i}^{p_{i+1}} \varphi'(s) ds + \int_{p_n}^t \varphi'(s) ds \\ &\geq \sum_{i=2}^{n-1} \int_{p_i}^{p_{i+1}} -\frac{b_i}{s} ds + \int_{p_n}^t -\frac{b_n}{t} ds \\ &= -\sum_{i=2}^{n-1} b_i \log(p_{i+1}/p_i) - b_n \log(t) + b_n \log(p_n) \\ &\geq -b_n \log(p_{n+1}) - b_n \log(t) \\ &\geq -2b_n \log(t). \end{aligned}$$

We will normalize φ so that $\varphi(p_2) = 0$. We observe that by making the p_i 's even bigger we can assume that $\lim_{t \rightarrow \infty} \varphi(t) = -\infty$. There is not loss of generality in assuming that. Finally define $u : \mathbb{R}^+ \rightarrow \mathbb{R}$ by the equation

$$u(t) = \log(t) + \varphi(t).$$

By definition of φ' we have that $u' = \frac{1}{t} + \varphi'(t) = (g_2^{-1/2})/t$, therefore u is surjective and strictly increasing (recall $g_2 \leq 2$). As commented at the beginning of this section, a function with the properties of u determine a function $T = T(u)$ and therefore a metric g_T . We now pick the reference point $o = (\mathbf{0}, u(1))$ and a parabolic isometry such that $d(o, po) = 1$. To help clarifying how we are using the properties required so far we will state a number of observations which end up in the proof of Theorem B.

Observation 1: The formula $K(u(t)) = -(g_2(t) - (t/2)g_2'(t))$, and property (3) in the definition of g_2 implies that $-2 \leq K \leq -1/3$. In particular the curvature of the metric g_T satisfies the pinching

$$-\frac{1}{9} \geq K_{g_T} \geq -4.$$

Observation 2: The hypothesis (4) in the definition of the sequence $\{p_n\}$ and the calculation of $J^*(t) - \frac{1}{2}t(J^*)'(t)$ on the intervals I_n gives us that

$$\lim_{t \rightarrow \infty} K(t) = -1.$$

For big enough t , $K(t)$ increases to -1 . In particular the function U is going to zero as t goes to infinity.

Observation 3: By property (4) in the definition of g_2 we know that

$$K(u(t)) \leq -a_{n+1},$$

for times $t \leq p_{n+1}$. Combining this and Remark 2 we get that for every $t < u(p_{n+1})$ we have $U(t) \leq -\frac{1}{n}$.

Observation 4: Using hypothesis (3) in the definition of the sequence $\{p_n\}$ and the lower bound for $\varphi(t)$ we get

$$\begin{aligned} \log(k(n+1)) &< (1 - 2b_n) \log(p_{n+1}) \\ &< \log(p_{n+1}) + \varphi(p_{n+1}) = u(p_{n+1}). \end{aligned}$$

Lemma 3. *There exists $m \in \mathbb{N}$ such that for every $n \geq k(m)$ and $|k| \leq k(n+1)$, the function U is at most $-\frac{1}{n}$ over the geodesic segment $[o, p^k o]$.*

Proof. It follows from **Obs. 1** that if $|k| \leq k(n+1)$, then the height of $[o, p^k o]$ is at most $u(k(n+1)) + D$. For n big enough this is less than $\log(k(n+1))$. Using **Obs. 4** we conclude that if $|k| \leq k(n+1)$, then the height of $[o, p^k o]$ is at most $u(p_{n+1})$. Finally **Obs. 3** gives us that $U \leq -\frac{1}{n}$ over the segment $[o, p^k o]$ if $|k| \leq k(n+1)$. \square

Lemma 4. *The critical exponent of $\mathcal{P} = \langle p \rangle$ is equal to $1/2$. Moreover \mathcal{P} is of divergence type.*

Proof. As explained in [DOP, Section 3], the Poincaré series of \mathcal{P} for the metric g_T is equivalent to the series

$$\sum_{n \in \mathbb{Z}} \exp(-2su(|n|)).$$

Observe that for every $\epsilon > 0$, there exists a natural number N such that if $n \geq N$, then we have $\varphi(n) > -\epsilon \log(t)$. In particular

$$\sum_{n \geq N} \exp(-2su(|n|)) \leq \sum_{n \geq N} \exp(-2s(1 - \epsilon) \log(|n|)).$$

If $s(1 - \epsilon) > 1/2$, then the right hand side converges. This implies that for every $\epsilon > 0$ the following inequality holds: $\delta_{\mathcal{P}} \leq \frac{1}{2(1-\epsilon)}$. On the other hand for big n we have $u(n) < \log(n)$. Then

$$\sum_{n > N'} \exp(-2s \log(|n|)) < \sum_{n > N'} \exp(-2su(|n|)).$$

Since the critical exponent of the LHS is $1/2$ (and of divergence type) we get the inequality $\delta_{\mathcal{P}} \geq 1/2$. We conclude that $\delta_{\mathcal{P}} = 1/2$ and \mathcal{P} is of divergence type. \square

Lemma 5. *The pair (\mathcal{P}, U) is of convergence type with respect to the metric g_T .*

Proof. We denote by d_T the distance function induced by g_T . Combining Lemma 3 and the definition of $k(n)$ we get the bounds

$$\begin{aligned} & \sum_{|k| \geq k(n)} \exp\left(\int_o^{p^k o} U - \frac{1}{2} d_T(o, p^k o)\right) \\ &= \sum_{n \geq m} \sum_{k=k(n)}^{k(n+1)-1} \exp\left(\int_o^{p^k o} U - \frac{1}{2} d_T(o, p^k o)\right) \\ &\leq \sum_{n \geq m} \sum_{k=k(n)}^{k(n+1)-1} \exp\left(-\left(\frac{1}{n} + \frac{1}{2}\right) d_T(o, p^k o)\right) \\ &\leq \sum_{n \geq m} \sum_{k=k(n)}^{k(n+1)-1} \exp\left(-\left(\frac{1}{n} + \frac{1}{2}\right) (2u(|k|))\right) \exp\left(\left(\frac{1}{n} + \frac{1}{2}\right) D\right) \\ &\leq \sum_{k=1} \frac{1}{k^2}. \end{aligned}$$

Obs. 2 and Lemma 2 implies that $\delta_{\mathcal{P}} = \delta_{\mathcal{P}}^U$ and then by Lemma 4 we know that the series above is exactly the Poincaré series associated to (\mathcal{P}, U) . This finish the proof of the Lemma. \square

4.2. Construction of the family $\{X_{n,m}\}$.

We now have all the ingredients to construct the Riemannian manifold announced in Theorem B. We will use the notation introduced at the beginning of this section. We start with (\mathbb{R}^N, g_{hyp}) , where g_{hyp} is the hyperbolic metric, and the function u constructed in Subsection 4.1. We choose a hyperbolic isometry h (for the hyperbolic metric) so that $H = \langle h \rangle$ is in Schottky position with respect to $\mathcal{P} = \langle p \rangle$. We moreover assume that the axis of h has height smaller than $u(1/2)$. Define Γ as the group generated by p and h and let $X = \mathbb{R}^N / \Gamma$. The closed geodesic associated to h is denoted by $\gamma = \gamma_h$. We can ‘cut’ the cusp associated to \mathcal{P} above height $u(1/2)$ and replace it by the cusp endowed with the metric g_T . This is possible because g_T is the hyperbolic metric on the region $\{(\mathbf{x}, t) : t < u(1)\}$. We have constructed a new Riemannian metric g on X . We lift the metric to the universal cover (\tilde{X}, g) , this is our Hadamard manifold. We will check that the Riemannian manifold (X, g) satisfies the properties announced in Theorem B. Since the geometric structure has change we will be careful with our notation. The generator of the parabolic subgroup of $Iso(\tilde{X}, g)$ corresponding to the cusp is denoted by p_* and h_* is the hyperbolic isometry associated to the closed geodesic γ in X . The

group generated by p_* is denoted by \mathcal{P}_* and the one generated by h_* is denoted by H_* . Let U_* be the normalization of the unstable jacobian of (X, g) as in Definition 10. As before, we will organize the relevant information in a couple of observations and lemmas. It will be convenient to define $F = -U_*$. We start with the following definition.

Definition 11. *We denote by $\Gamma_{n,m}$ to the group generated by h_*^n and p_*^m . Let $X_{n,m}$ be the covering of X associated to the subgroup $\Gamma_{n,m}$ of Γ . The lift of a potential G on X to $X_{n,m}$ is denoted by $G_{n,m}$.*

Observation 5: The closed geodesic γ lies in the region where g is hyperbolic. This implies that U_* and F vanish along γ .

Observation 6: Since F^{su} is locally defined in terms of g and every local structure is preserved under taking coverings, we conclude that the potential $F_{n,m}^{su}$ corresponds to the geometric potential of the metric g lifted to $X_{n,m}$.

Lemma 6. *The pair $(\mathcal{P}_*, -tF)$ is of convergence type for every $t \geq 1$.*

Proof. The reference point used in the proof of Lemma 5 lies in the piece of X coming from (\mathbb{R}^N, g_T) . Since horoballs are convex it follows that $d_T = d$ on that region. This implies that the series estimated in Lemma 5 is exactly the Poincaré series associated to \mathcal{P}_* , which implies that $(\mathcal{P}_*, -F)$ is of convergence type. By the construction of g_2 we know that F is positive above height $u(2)$. Change the reference point to o' lying above height $u(2)$. By convexity of this region and the definition of g_2 we get that $-t \int_{o'}^{p^k o'} F \leq \int_{o'}^{p^k o'} F$, for every $t \geq 1$. Plugging this into the Poincaré series of $-tF$ and $-F$ implies the lemma. \square

The idea now is to apply Theorem 6 to the potential $F = -U_*$. Lemma 6 and **Obs. 2** implies that F belongs to the class \mathcal{F}_s . Despite F is not nonnegative, by **Obs. 5** and the discussion at the end of Section 3, we know that F_n exhibits a phase transition for sufficiently big n . This is not a very satisfactory answer, one would like to know which type of phase transition we have, either a type A or type B phase transition. By the construction of the modified metric at the cusp, a phase transition of type B seems unlikely to occur. Since we are not able to completely rule out that case, we present an argument to justify that a type A phase transition is always possible to construct. As before, we denote by \tilde{F} to the lift of F to the universal cover of X . Denote by $\xi \in \partial_\infty \tilde{X}$ to the parabolic fixed point of \mathcal{P}_* . By construction of g_2 we know that there exist real numbers s and L such that \tilde{F} is negative precisely at vectors whose base lies in the interior of $B_\xi(s) \setminus B_\xi(s+L)$ and it is zero for vectors with base in $X \setminus B_\xi(s)$. It follows easily from this that there exists m_0 such that if $m \geq m_0$, then

$$(2) \quad \int_o^{g \cdot o} \tilde{F} > 0,$$

for every $g \in \Gamma_{1,m}$, in particular for every $g \in \Gamma_{n,m}$. It follows from the definition of the Poincaré series and the fact that F goes to zero through the cusp of X (see **Obs. 2**), that if the pair $(\Gamma_{n,1}, -F_{n,1})$ is of convergence type with $\delta_{\Gamma_{n,1}}^{-F_{n,1}} = \delta_{\mathcal{P}}$, then the same holds for $(\Gamma_{n,m}, -F_{n,m})$. Moreover, if $m \geq m_0$, then inequality (2) and the definition of the Poincaré series implies that if $(\Gamma_{n,m}, -F_{n,m})$ is of convergence

type with $\delta_{\Gamma_{n,m}}^{-F_{n,m}} = \delta_{\mathcal{P}}$, then the same holds for $(\Gamma_{n,m}, -tF_{n,m})$ for every $t \geq 1$. Combining **Obs. 6** and the discussion above we conclude that $F_{n,m}$ admits a phase transition of type A for big enough n and m .

4.3. Conclusions.

In Subsection 4.2 we constructed a family of geometrically finite negatively curved manifolds $(X_{n,m}, g)$ (see Definition 11) for which $F_{n,m}$ exhibits a phase transition of type A if n and m are big enough. Since

$$tF_{n,m} = -tF_{n,m}^{su} - t(N-1),$$

the existence of an equilibrium state for $tF_{n,m}$ is equivalent to the existence of one for $-tF_{n,m}^{su}$. Just as in Theorem 6, there exists $t_{n,m} \in [-1, 0)$ such that $tF_{n,m}^{su}$ has an equilibrium state for all $t > t_{n,m}$ and there is not equilibrium state for $t < t_{n,m}$. The behaviour at $t = t_{n,m}$ is more subtle and we are not able to decide the existence (or not) of an equilibrium measure. Since by construction we can not ensure $t_{n,m} \neq -1$, we can not immediately decide if $-F_{n,m}^{su}$ has or not an equilibrium state. This difficulty can be overcome by defining $a'_n = a_{2n}$ and doing the substitution $a_n \mapsto a'_n$. It is clear from the construction that $X_{n,m}$ are extended Schottky manifolds. By Lemma 4 we know that \mathcal{P}_* and $\langle p_*^n \rangle$ are of divergence type, then using Theorem 3 we have $\delta_{\langle p_*^n \rangle} < \delta_{\Gamma_{n,m}}$. We conclude that the geodesic flow on $X_{n,m}$ has a probability measure of maximal entropy. All this together proves Theorem B.

Theorem 7. *There exist a family of extended Schottky manifolds for which the geometric potential exhibits a phase transition as in Figure 4. The geodesic flow on X has an unique measure of maximal entropy.*

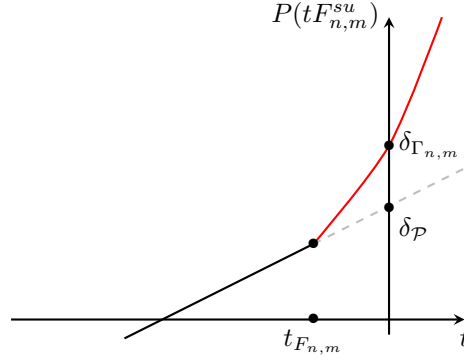


Figure 4: Phase transition for $P(tF_{n,m}^{su})$

As mentioned in Remark 3 we could have done the same construction for an extended Schottky manifold with one parabolic and arbitrary number of hyperbolic generators (modulo taking big powers of the parabolic and hyperbolic generators). Combining Remark 6 and Theorem 7 we obtain the following result.

Corollary 3. *Let X be a M -punctured sphere with $M \geq 3$. We can endow X with a complete Riemannian metric g with negative sectional curvature having exactly one cusp, such that the geometric potential exhibits a phase transition as in Figure 4. Moreover, the geodesic flow of (X, g) has an unique measure of maximal entropy and g is hyperbolic outside a neighborhood of the cusp.*

REFERENCES

- [BI] L. Barreira, G. Iommi, *Multifractal analysis and phase transitions for hyperbolic and parabolic horseshoes*. Israel J. Math. **181** (2011), 347-379.
- [Bo] B. Bowditch, *Geometrical finiteness with variable negative curvature*. Duke Math. J. **77** (1995), no. 1, 229-274.
- [B] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*. Trans. Amer. Math. Soc. **153** (1971), 401-414.
- [BR] R. Bowen and D. Ruelle, *The ergodic theory of Axiom A flows*. Invent. Math. **29** (1975), 181-202.
- [Cou] Y. Coudène, *Gibbs measures on negatively curved manifolds*. J. Dynam. Control Syst. **9** (2003), 89-101.
- [DP] F. Dal'bo and M. Peigné, *Some negatively curved manifolds with cusps, mixing and counting*. J. Reine Angew. Math. **497** (1998), 141-169.
- [DOP] F. Dal'bo, J-P. Otal and M. Peigné, *Séries de Poincaré des groupes géométriquement finis*. Israel J. Math. **118** (2000), 109-124.
- [EKP] M. Einsiedler, S. Kadyrov, A. D. Pohl, *Escape of mass and entropy for diagonal flows in real rank one situations*, Israel J. Math. **210** (2015), no. 1, 245-295.
- [HK] M. Handel; B. Kitchens, *Metrics and entropy for non-compact spaces*. With an appendix by Daniel J. Rudolph. Israel J. Math. **91** (1995), no. 1-3, 253-271.
- [IJ] G. Iommi, T. Jordan, *Phase transitions for suspension flows*. Comm. Math. Phys. **320** (2013), no. 2, 475-498.
- [IRV] G. Iommi, F. Riquelme, A. Velozo, *Entropy in the cusp and phase transitions for geodesic flows*. to appear in Israel J. Math.
- [K] W. Klingenberg, *Riemannian geometry*, de Gruyter Studies in Mathematics, (1982).
- [OP] J-P. Otal and M. Peigné, *Principe variationnel et groupes kleinien*. Duke Math. J. **125** (2004), no. 1, 15-44.
- [P] M. Peigné, *On some exotic Schottky groups*. Discrete Contin. Dyn. Syst. **31** (2011), no.2, 559-579.
- [PPS] F. Paulin, M. Pollicott and B. Schapira *Equilibrium states in negative curvature*. Astérisque No. **373** (2015), viii+281 pp.
- [PZ] Y. Pesin, K. Zhang, *Phase transitions for uniformly expanding maps*. J. Stat. Phys. **122** (2006), no. 6, 1095-1110.
- [R] F. Riquelme, *Ruelle's inequality and Pesin's entropy formula for the geodesic flow on negatively curved noncompact manifolds*, arxiv:1601.02843 .
- [RV] F. Riquelme, A. Velozo, *Escape of mass and entropy for geodesic flows*. to appear in Erg. Th. and Dyn. Systems.
- [S] O. Sarig, *Phase transitions for countable Markov shifts*. Comm. Math. Phys. **217** (2001), no. 3, 555-577.
- [Su] D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions*. Publ. Math. I.H.E.S., **50** (1979), 171-202.

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